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A method for estimating upper and lower bounds of eigenvalues of closed-loop systems with uncertain parameters

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Abstract

Using the convex model theory, the vibration control problem of structures with uncertain parameters is discussed, which is approximated by a deterministic one. A method for estimating the upper and lower bounds of eigenvalues of the closed-loop systems is presented by combining the matrix perturbation and optimization. The present method is applied to a vibration system to illustrate the application. The numerical results show that the present method is effective.

1. Introduction

The vibration control theory for systems with deterministic parameters has been well developed. For example, Refs. [1–3] developed the standard methods for vibration control, and Refs. [4,5] discussed the modal controllability/ observability and modal optimal control for defective/near defective systems with repeated/close eigenvalues.

However, in actual situations, the structural parameters are often uncertain, such as the inaccuracy of the measurement, errors in the manufacturing process, invalidity of some components, etc. Therefore, the uncertain concept plays an important role in the control problem of the vibration structures. Many studies have been done about the control problems only from the viewpoint of mathematics. For example, Refs. [6,7] discussed the sufficient and necessary conditions of the dynamic stability for the uncertain systems; Refs. [8,9] discussed the robustness of control systems with uncertain parameters; Ref. [10] discussed the stability of an uncertain matrix.

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The most common methods for solving uncertainty problems are to model the structural parameters as a random vector. Unfortunately, the probabilistic approaches cannot give reliable results unless sufficient experimental data are available to validate the assumptions about the joint probability densities of the random variables or functions involved. Recently, the convex model has been used to deal with the uncertainty problems in robust analysis of control systems and structural failures. For instance, Ben-Haim and Elishakoff [11] and Lindberg [12] used the convex model to study the dynamic response and failure of structures with pulse loads. Shi and Gao used the convex model to solve the robustness of control system [13].

In this paper, the convex model is used to deal with the control problems of systems with uncertain parameters. The uncertainties of the structural parameters are described by an ellipsoid. The control problems of the uncertain systems are transformed into ones of the deterministic systems. At first, by using the method of pole allocation, the state feedback gain matrix of the systems with deterministic parameters can be obtained, and then it is applied into the actual uncertain systems. By combining the convex model of the parameters with the perturbation method, the expressions for estimating the upper and lower bounds of the real and imaginary parts of the eigenvalues of the closed-loop systems can be developed. A numerical example is given to illustrate the application of the approach presented in this study.

2. The definition of the problem

Consider the linear vibration control equation in state space

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t). \tag{1}$$

By using the state feedback law, the input vector is

$$\mathbf{u}(t) = \mathbf{G}\mathbf{x}(t), \tag{2}$$

where $\mathbf{x}(t)$ is the $2n \times 1$ state vector, $\mathbf{u}(t)$ is an $m \times 1$ input vector, $\mathbf{A}$ is the $2n \times 2n$ state matrix, $\mathbf{B}$ is a $2n \times m$ input coefficient matrix, $\mathbf{G}$ is an $m \times 2n$ state feedback gain matrix.

The state matrix $\mathbf{A}$ and input coefficient matrix $\mathbf{B}$ of the uncertain system can be expressed as

$$\begin{aligned}\mathbf{A} &= \mathbf{A}_0 + \Delta\mathbf{A},\\ \mathbf{B} &= \mathbf{B}_0 + \Delta\mathbf{B},\end{aligned} \tag{3}$$

where $\mathbf{A}_0$ and $\mathbf{B}_0$ are the deterministic parts of the state matrix and the input coefficient matrix, respectively; $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ are the uncertain parts of the state matrix and the input coefficient matrix, respectively. Correspondingly, the uncertain state vector $\mathbf{x}$, the uncertain gain matrix $\mathbf{G}$ and the uncertain input vector $\mathbf{u}$ are

$$\begin{aligned}\mathbf{x} &= \mathbf{x}_0 + \Delta\mathbf{x},\\ \mathbf{u} &= \mathbf{u}_0 + \Delta\mathbf{u},\\ \mathbf{G} &= \mathbf{G}_0 + \Delta\mathbf{G},\end{aligned} \tag{4}$$

where $\mathbf{x}_0$, $\mathbf{u}_0$ and $\mathbf{G}_0$ are the deterministic parts of the state vector, the input vector and the gain matrix. $\Delta\mathbf{x}$, $\Delta\mathbf{u}$ and $\Delta\mathbf{G}$ are their uncertain parts, respectively.

Substituting Eqs. (3) and (4) into Eqs. (1) and (2) yields

$$\dot{\mathbf{x}}_0 + \Delta\dot{\mathbf{x}} = (\mathbf{A}_0 + \Delta\mathbf{A})(\mathbf{x}_0 + \Delta\mathbf{x}) + (\mathbf{B}_0 + \Delta\mathbf{B})(\mathbf{u}_0 + \Delta\mathbf{u}) \tag{5}$$

and

$$\mathbf{u}_0 + \Delta\mathbf{u} = (\mathbf{G}_0 + \Delta\mathbf{G})(\mathbf{x}_0 + \Delta\mathbf{x}). \tag{6}$$

Expanding Eqs. (5) and (6), we have

$$\dot{\mathbf{x}}_0 + \Delta\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x}_0 + \mathbf{A}_0\Delta\mathbf{x} + \Delta\mathbf{A}\mathbf{x}_0 + \Delta\mathbf{A}\Delta\mathbf{x} + \mathbf{B}_0\mathbf{u}_0 + \mathbf{B}_0\Delta\mathbf{u} + \Delta\mathbf{B}\mathbf{u}_0 + \Delta\mathbf{B}\Delta\mathbf{u} \tag{7}$$

and

$$\mathbf{u}_0 + \Delta\mathbf{u} = \mathbf{G}_0\mathbf{x}_0 + \mathbf{G}_0\Delta\mathbf{x} + \Delta\mathbf{G}\mathbf{x}_0 + \Delta\mathbf{G}\Delta\mathbf{x}. \tag{8}$$

Neglecting the higher orders of the above Eqs. (7) and (8), and equating the coefficients of the same orders of the left and the right sides, we obtain

$$\begin{aligned} \dot{\mathbf{x}}_0 &= \mathbf{A}_0\mathbf{x}_0 + \mathbf{B}_0\mathbf{u}_0, \\ \mathbf{u}_0 &= \mathbf{G}_0\mathbf{x}_0 \end{aligned} \tag{9}$$

and

$$\begin{aligned} \Delta\dot{\mathbf{x}} &= \mathbf{A}_0\Delta\mathbf{x} + \Delta\mathbf{A}\mathbf{x}_0 + \mathbf{B}_0\Delta\mathbf{u} + \Delta\mathbf{B}\mathbf{u}_0, \\ \Delta\mathbf{u} &= \mathbf{G}_0\Delta\mathbf{x} + \Delta\mathbf{G}\mathbf{x}_0. \end{aligned} \tag{10}$$

From the above discussion it can be seen that the uncertain system (1) and (2) have been separated into the deterministic part (9) and the uncertain part (10). The closed-loop system corresponding to the deterministic system (9) is

$$\dot{\mathbf{x}}_0(t) = (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)\mathbf{x}_0(t) \tag{11}$$

and the corresponding eigenvalue problem is

$$\lambda_0\mathbf{u}_0 = (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)\mathbf{u}_0. \tag{12}$$

3. The gain matrix of the deterministic system [3,14]

In the pole allocation method, to guarantee asymptotic stability, the closed-loop poles can be selected in advance and the gains are determined so as to produce these poles. Thus, when the closed-loop eigenvalues of Eq. (11) are assigned to be $\lambda_1^*, \lambda_2^*, \ldots, \lambda_{2n}^*$, by using the pole allocation, the gain matrix $\mathbf{G}_0$ of the deterministic system (9) can be determined.

First, we transform Eq. (9) into the control equation in modal co-ordinates. Suppose the left and the right modal matrices $\mathbf{U}_0 = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{2n}]$ and $\mathbf{V}_0 = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n}]$ have been obtained. They satisfy the following equations:

$$\mathbf{V}_0^{\mathrm{T}}\mathbf{A}_0\mathbf{U}_0 = \mathbf{\Lambda}_0, \quad \mathbf{V}_0^{\mathrm{T}}\mathbf{U}_0 = \mathbf{I}, \tag{13}$$

where $\mathbf{\Lambda}_0 = diag(\lambda_{01}, \lambda_{02}, \ldots, \lambda_{02n})$ is the diagonal matrix of the eigenvalues of the deterministic system.

With the modal transformation

$$\mathbf{x}_0(t) = \mathbf{U}_0\boldsymbol{\xi}(t), \tag{14}$$

the Eq. (9) can be transferred into

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{\Lambda}_0\boldsymbol{\xi}(t) + \mathbf{B}_0'\mathbf{u}_0(t) \tag{15}$$

and

$$\mathbf{u}_0(t) = \mathbf{G}_0'\boldsymbol{\xi}(t). \tag{16}$$

If the single input is used, $\mathbf{B}_0$ is a column vector, $\mathbf{G}_0$ is a row vector. Let

$$\mathbf{B}_0' = \mathbf{V}_0^{\mathrm{T}}\mathbf{B}_0 = (b_1', b_2', \ldots, b_{2n}')^{\mathrm{T}}, \quad \mathbf{G}_0' = \mathbf{G}_0\mathbf{U}_0 = (g_1', g_2', \ldots g_{2n}') \tag{17}$$

and substituting Eq. (16) into Eq. (15), one has

$$\dot{\boldsymbol{\xi}}(t) = (\boldsymbol{\Lambda}_0 + \mathbf{B}_0'\mathbf{G}_0')\boldsymbol{\xi}(t). \tag{18}$$

In Eq. (18), suppose the assigned eigenvalues are λ_i^* $(i = 1, 2\ldots, 2n)$, the corresponding eigenvectors are $\mathbf{w}_i$ $(i = 1, 2\ldots, 2n)$, and they satisfy the following eigenproblem:

$$(\boldsymbol{\Lambda}_0 + \mathbf{B}_0'\mathbf{G}_0')\mathbf{w}_i = \lambda_i^*\mathbf{w}_i \quad (i = 1, 2, \ldots, 2n) \tag{19}$$

i.e.

$$(\boldsymbol{\Lambda}_0 + \mathbf{B}_0'\mathbf{G}_0' - \lambda_i^*\mathbf{I})\mathbf{w}_i = \mathbf{0} \quad (i = 1, 2, \ldots, 2n). \tag{20}$$

Because $\mathbf{w}_i \neq 0$, then there exists

$$det(\boldsymbol{\Lambda}_0 + \mathbf{B}_0'\mathbf{G}_0' - \lambda_i^*\mathbf{I}) = 0. \tag{21}$$

Solving Eq. (21), we obtain

$$g_i' = \frac{\prod_{k=1}^{2n}(\lambda_k^* - \lambda_{0i})}{b_i' \prod_{\substack{k=1 \\ k\neq i}}^{2n}(\lambda_{0k} - \lambda_{0i})}, \quad i = 1, 2, \ldots, 2n \tag{22}$$

thus obtaining the matrix $\mathbf{G}_0' = (g_1', g_2', \ldots, g_{2n}')$.

From Eq. (14), we obtain

$$\boldsymbol{\xi}(t) = \mathbf{V}_0^{\mathrm{T}}\mathbf{x}_0(t). \tag{23}$$

Substituting Eq. (23) into Eq. (16) yields

$$\begin{aligned} u_0(t) &= \mathbf{G}_0'\mathbf{V}_0^{\mathrm{T}}\mathbf{x}_0(t) \\ &= \mathbf{G}_0\mathbf{x}_0(t), \end{aligned} \tag{24}$$

where

$$\mathbf{G}_0 = \mathbf{G}_0'\mathbf{V}_0^{\mathrm{T}}. \tag{25}$$

If the deterministic gain matrix $\mathbf{G}_0$ is applied to the uncertain system, there must exist some errors between the closed-loop eigenvalues and the assigned eigenvalues λ_i^* $(i = 1, 2, \ldots, 2n)$. By combining the convex model of the uncertain parameters with the perturbation method, the expressions for computing the upper and lower bounds of the closed-loop eigenvalues λ_i $(i = 1, 2, \ldots, 2n)$ can be developed.

4. The convex model theory [11,15]

The method of describing the uncertainties by convex set is called convex model, which does not need precise information and is used broadly.

If the uncertainty $\boldsymbol{\alpha}$ is confined to a convex set Ω, i.e. $\boldsymbol{\alpha} \in \Omega$, where the elements of the vector $\boldsymbol{\alpha}$ are values or functions. Thus Ω is the convex model of the uncertainty $\boldsymbol{\alpha}$. The several common-used convex models are listed as follows:

(1) *The maximum bound convex model*:

$$\Omega_{MB} = \{\boldsymbol{\alpha}(t) \in R^r : |\alpha_j(t)| \leqslant \bar{\alpha}_j,\ j = 1, \ldots, r\}, \tag{26}$$

where $\bar{\alpha}_j,\ j = 1, \ldots, r$ are constants.

(2) *The integral energy bound convex model*:

$$\Omega_{IEB} = \left\{\boldsymbol{\alpha}(t) \in R^r : \int_0^T \boldsymbol{\alpha}^{\mathrm{T}}(\tau)\boldsymbol{\alpha}(\tau)\,\mathrm{d}\tau \leqslant E^2\right\}, \tag{27}$$

where E is a positive real constant.

(3) *The local energy bound convex model*:

$$\Omega_{LEB} = \{\boldsymbol{\alpha}(t) \in R^r : [\boldsymbol{\alpha}(t) - \bar{\boldsymbol{\alpha}}(t)]^{\mathrm{T}}[\boldsymbol{\alpha}(t) - \bar{\boldsymbol{\alpha}}(t)] \leqslant \rho^2(t)\}, \tag{28}$$

where $\bar{\boldsymbol{\alpha}}(t)$ is a vector function, and $\rho^2(t)$ is a energy bound.

(4) *The ellipsoidal convex model*:

$$\Omega_{ELP} = \{\boldsymbol{\alpha}(t) \in R^r : \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}\boldsymbol{\alpha} \leqslant \theta^2\}, \tag{29}$$

where $\boldsymbol{\alpha}$ is the uncertain parameter vector, $\mathbf{W}$ is the symmetric positive weighted matrix, θ is a given positive real constant. The convex model means that all the uncertain parameters are constrained into the N-dimension ellipsoid.

In the following, the uncertainties of the parameters are described by Ellipsoid Convex Model (29). By combining the convex model of uncertainties with perturbation theory [15] to estimate the upper and lower bounds of the real and imaginary parts of closed-loop eigenvalues of the actual uncertain system is developed.

5. Upper and lower bounds of eigenvalues of the closed-loop systems

According to the convex model theory presented above, if the uncertain parameters are denoted by α_j, the uncertain state matrix can be written as

$$\mathbf{A} = \mathbf{A}_0 + \sum_{j=1}^{m} \alpha_j \mathbf{A}_j, \tag{30}$$

where $\mathbf{A}_0$ is the state matrix with deterministic parameters; m is the number of uncertain parameters; $\mathbf{A}_j$ is the jth state sub-matrix. According to the convex model theory, α_j satisfy

$$\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}\boldsymbol{\alpha} \leqslant \theta^2, \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)^{\mathrm{T}}, \tag{31}$$

where θ is a given positive real constant; $\mathbf{W}$ is a symmetric positive weighted matrix; $\boldsymbol{\alpha}$ is a real constant vector.

Applying the state feedback gain matrix obtained above to the system with uncertainties, then the closed-loop system is

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A}_0 + \sum_{j=1}^{m} \alpha_j \mathbf{A}_j + \mathbf{B}\mathbf{G}_0\right)\mathbf{x}(t). \tag{32}$$

Here, we assume the input coefficient matrix is deterministic. i.e., $\mathbf{B}_0 = \mathbf{B}$.

By using the following notations:

$$\begin{aligned} &\mathbf{A}_0 + \mathbf{B}\mathbf{G}_0 = \mathbf{C}, \\ &\sum_{j=1}^{m} \alpha_j \mathbf{A}_j = \Delta\mathbf{C}, \end{aligned} \tag{33}$$

Eq. (32) becomes

$$\dot{\mathbf{x}}(t) = (\mathbf{C} + \Delta\mathbf{C})\mathbf{x}(t), \tag{34}$$

From Section 3, we know that the eigenvalues of the matrix $\mathbf{C}$ are the assigned eigenvalues λ_i^* $(i = 1, 2, \ldots 2n)$. And the corresponding eigenvalue problem is

$$\mathbf{C}\mathbf{\Phi} = \mathbf{\Phi}\mathbf{\Lambda}^*, \quad \mathbf{C}^{\mathrm{T}}\mathbf{\Psi} = \mathbf{\Psi}\mathbf{\Lambda}^*. \tag{35}$$

In engineering design, the situation involving small uncertainties is often considered. According to the perturbation theory [16], the eigenvalues of the closed-loop system can be expressed as

$$\begin{aligned} \lambda_i &= \lambda_i^* + \psi_i^{\mathrm{T}}(\Delta\mathbf{C})\varphi_i \\ &= \lambda_i^* + \psi_i^{\mathrm{T}}\left(\sum_{j=1}^{m} \alpha_j \mathbf{A}_j\right)\varphi_i \\ &= \lambda_i^* + \sum_{j=1}^{m} \alpha_j(\psi_i^{\mathrm{T}}\mathbf{A}_j\varphi_i) \\ &= \lambda_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{A}^i \quad (i = 1, 2 \ldots, 2n), \end{aligned} \tag{36}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)^{\mathrm{T}}$, $\mathbf{A}^i = (\psi_i^{\mathrm{T}}\mathbf{A}_1\varphi_i, \psi_i^{\mathrm{T}}\mathbf{A}_2\varphi_i, \ldots, \psi_i^{\mathrm{T}}\mathbf{A}_m\varphi_i)^{\mathrm{T}}$; ψ_i and φ_i are the ith left and the ith right modal vectors of $\mathbf{C}$, respectively.

Because the eigenvalues of the system are complex, the upper and lower bounds of the real and the imaginary parts of the closed-loop eigenvalues will be discussed, respectively.

Suppose

$$\begin{aligned} &\lambda_i = d_i + f_i\mathrm{j}, \\ &\lambda_i^* = d_i^* + f_i^*\mathrm{j}, \quad i = 1, 2, \ldots, 2n, \ \mathrm{j} = \sqrt{-1}, \\ &\mathbf{A}^i = \mathbf{D}_i + \mathbf{F}_i\mathrm{j}, \end{aligned} \tag{37}$$

where d_i and f_i are the real and imaginary parts of λ_i, respectively; d_i^* and f_i^* are the real and imaginary parts of λ_i^*, respectively; $\mathbf{D}_i$ and $\mathbf{F}_i$ are the real and imaginary parts of $\mathbf{A}^i$, respectively.

Substituting Eq. (37) into Eq. (36) yields

$$\begin{aligned} d_i + f_i\mathrm{j} &= d_i^* + f_i^*\mathrm{j} + \boldsymbol{\alpha}^{\mathrm{T}}(\mathbf{D}_i + \mathbf{F}_i\mathrm{j}) \\ &= d_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{D}_i + (f_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{F}_i)\mathrm{j}. \end{aligned} \tag{38}$$

By letting the real and imaginary parts of the left side and equating the counterparts of the right side of Eq. (38), respectively, we can obtain

$$d_i = d_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{D}_i, \quad i = 1, 2, \ldots 2n, \tag{39}$$

$$f_i = f_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{F}_i, \quad i = 1, 2, \ldots 2n. \tag{40}$$

In the following, the expressions of the extremums of the real and imaginary parts of the closed-loop eigenvalues will be given.

When the uncertain parameters, $\boldsymbol{\alpha}$, vary including the bound described by the ellipse (31), by using the technique of optimization, the approximate extremums of the real and imaginary parts of the eigenvalues can be determined.

$$\left.\begin{aligned} (d_i)_{max} &= max\{d_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{D}_i\}, \\ (d_i)_{min} &= min\{d_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{D}_i\}, \\ (f_i)_{max} &= max\{f_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{F}_i\}, \\ (f_i)_{min} &= min\{f_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{F}_i\}, \end{aligned}\right. \quad i = 1, 2, \ldots, 2n. \tag{41}$$

According to the convex model theory, the extremums of (39) and (40) will occur on the boundary of the ellipsoid described by Eq. (31) [17], we obtain

$$S(\theta, \mathbf{W}) = \{\boldsymbol{\alpha} : \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}\boldsymbol{\alpha} = \theta^2\}. \tag{42}$$

By using the Lagrange multiplier method, the Lagrangian functions can be obtained

$$H_1 = d_i + t_1(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}\boldsymbol{\alpha} - \theta^2) = d_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{D}_i + t_1(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}\boldsymbol{\alpha} - \theta^2), \tag{43}$$

$$H_2 = f_i + t_2(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}\boldsymbol{\alpha} - \theta^2) = f_i^* + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{F}_i + t_2(\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{W}\boldsymbol{\alpha} - \theta^2), \tag{44}$$

where t_1 and t_2 are the Lagrange multipliers.

The necessary conditions for the extremums of H_1 and H_2 are

$$\begin{aligned} \frac{\partial H_1}{\partial \boldsymbol{\alpha}} &= \mathbf{D}_i + 2t_1\mathbf{W}\boldsymbol{\alpha} = 0, \\ \frac{\partial H_2}{\partial \boldsymbol{\alpha}} &= \mathbf{F}_i + 2t_2\mathbf{W}\boldsymbol{\alpha} = 0. \end{aligned} \tag{45}$$

Hence we have

$$\boldsymbol{\alpha} = -\frac{\mathbf{W}^{-1}\mathbf{D}_i}{2t_1}, \tag{46}$$

$$\boldsymbol{\alpha} = -\frac{\mathbf{W}^{-1}\mathbf{F}_i}{2t_2}. \tag{47}$$

Substituting Eqs. (46) and (47) into Eq. (42) yields

$$2t_1 = \pm \frac{\sqrt{\mathbf{D}_i^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{D}_i}}{\theta}, \tag{48}$$

$$2t_2 = \pm \frac{\sqrt{\mathbf{F}_i^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{F}_i}}{\theta}. \tag{49}$$

Substituting Eq. (48) into Eq. (46), $\boldsymbol{\alpha}$ can be obtained when the real parts of the eigenvalues take the extremums, i.e.

$$\boldsymbol{\alpha} = \pm \frac{\theta\mathbf{W}^{-1}\mathbf{D}_i}{\sqrt{\mathbf{D}_i^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{D}_i}}. \tag{50}$$

Substituting Eq. (50) into Eq. (39), The upper and lower bounds of the real parts of the eigenvalues can be obtained as follows:

$$\begin{aligned}(d_i)_{max} &= d_i^* + \theta\sqrt{\mathbf{D}_i^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{D}_i},\\ (d_i)_{min} &= d_i^* - \theta\sqrt{\mathbf{D}_i^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{D}_i}, \quad i = 1, 2, \dots, 2n.\end{aligned} \tag{51}$$

The similar expressions for the imaginary parts of the eigenvalues can be obtained as follows:

$$\begin{aligned}(f_i)_{max} &= f_i^* + \theta\sqrt{\mathbf{F}_i^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{F}_i},\\ (f_i)_{min} &= f_i^* - \theta\sqrt{\mathbf{F}_i^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{F}_i}, \quad i = 1, 2, \dots, 2n.\end{aligned} \tag{52}$$

And the corresponding $\boldsymbol{\alpha}$ is

$$\boldsymbol{\alpha} = \pm \frac{\theta\mathbf{W}^{-1}\mathbf{F}_i}{\sqrt{\mathbf{F}_i^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{F}_i}}. \tag{53}$$

From Eqs. (51) and (52), it can be seen that the uncertainties of the real and imaginary parts of the eigenvalues will increase as the uncertainties of the parameters increase.

6. Numerical example

In order to illustrate the application of the present method, a numerical example is given as follows.

Consider a vibration control system shown in Fig. 1. An input force is imposed on m_2. Assume that the mass coefficients m_1 and m_2 are deterministic, and the stiffness coefficients of springs, k_1 and k_2, have some errors in the manufacturing process. k_1 and k_2 can be expressed as $k_1 = (1 + \alpha_1)k$, $k_2 = (1 + \alpha_2)k$, where k is a constant, α_1 and α_2 are used to describe the errors, i.e., the uncertain parameters in Eqs. (29) and (31). Assume $m_1 = 1$, $m_2 = 2$, and $k = 1$.

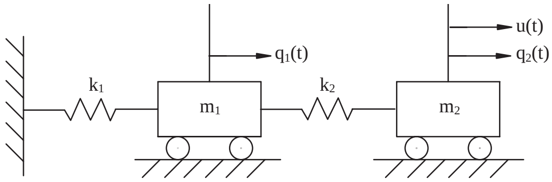

Fig. 1. The vibration control system.

The mass matrix is deterministic

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The stiffness matrix of the system with uncertain parameters is

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} + \alpha_1 \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$
$$= \mathbf{K}_0 + \alpha_1 \mathbf{K}_1 + \alpha_2 \mathbf{K}_2,$$

where

$$\mathbf{K}_0 = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}.$$

Suppose the state vector is

$$\mathbf{x}(t) = \begin{bmatrix} q_1(t) & q_2(t) & \dot{q}_1(t) & \dot{q}_2(t) \end{bmatrix}^{\mathrm{T}}.$$

Then the state matrix of the system is

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_0 & \mathbf{0} \end{bmatrix} + \alpha_1 \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_1 & \mathbf{0} \end{bmatrix} + \alpha_2 \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_2 & \mathbf{0} \end{bmatrix}$$
$$= \mathbf{A}_0 + \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2,$$

where

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_0 & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_1 & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_2 & \mathbf{0} \end{bmatrix}.$$

The state matrix with uncertain parameters can be expressed as

$$\mathbf{A} = \mathbf{A}_0 + \alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2,$$

where $\mathbf{A}_0$ is the state matrix with deterministic parameters, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the state sub-matrices corresponding to the uncertain parameters α_1 and α_2, respectively.

In the computation, we assume that

$$\mathbf{W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

i.e.

$$\alpha_1^2 + \alpha_2^2 \leqslant \theta^2.$$

The eigenvalues of $\mathbf{A}_0$ are

$$\lambda_{01} = 1.51022i, \quad \lambda_{02} = -1.51022i, \quad \lambda_{03} = 0.46821i, \quad \lambda_{04} = -0.46821i.$$

If the frequencies of the system are unchanged, and only the damping of the system is assigned, that is the real parts of the eigenvalues of the system can be assigned as −0.50000. Using Eq. (25), the state feedback gain matrix for the system with deterministic parameters can be obtained

$$\mathbf{G}_0 = \begin{bmatrix} 4.62500 & -3.00000 & 2.00000 & -4.00000 \end{bmatrix}.$$

If $\mathbf{G}_0$, the feedback gain matrix, is applied to the actual system with uncertain parameters, the closed-loop eigenvalues will has some perturbations. Using Eqs. (51) and (52), the upper and lower bounds of the real and imaginary parts of the eigenvalues of the closed-loop system with uncertain parameters are obtained and listed in Tables 1 and 2 with different values of θ, where $R(\lambda_{1,2})$ denotes the real part of the first and second eigenvalues; $I(\lambda_{1,2})$ the imaginary part of the first and second eigenvalues; $R(\lambda_{3,4})$ the real part of the third and fourth eigenvalues; $I(\lambda_{3,4})$ the imaginary part of the third and fourth eigenvalues.

The curves of upper and lower bounds of eigenvalues are shown in Figs. 2–5 where $R(\lambda)_L$, $R(\lambda)_U$ are the lower and upper bounds of the real parts of eigenvalues, respectively; and $I(\lambda)_L$, $I(\lambda)_U$ the lower and upper bounds of the imaginary parts of eigenvalues, respectively; $R(\lambda)_0$ and $I(\lambda)_0$ the real and imaginary parts of eigenvalues of the system with deterministic parameters, respectively. From Figs. 2–5, it can be seen that the relative errors will be large as the uncertainty of parameters, θ, increases. For instance, if $\theta = 0.01$, the max relative error is 2.84204% at the

Table 1
The upper and lower bounds of the real and imaginary parts of the eigenvalues of the closed-loop system with uncertain parameters ($\theta = 0.01$)

	Average	Upper bounds	Lower bounds	Errors (%)
$R(\lambda_{1,2})$	−0.50000	−0.49729	−0.50271	1.08465
$I(\lambda_{1,2})$	1.51022	1.51664	1.50381	0.84943
$R(\lambda_{3,4})$	−0.50000	−0.49729	−0.50271	1.08465
$I(\lambda_{3,4})$	0.46821	0.47487	0.46156	2.84204

Table 2
The upper and lower bounds of the real and imaginary parts of the eigenvalues of the closed-loop system with uncertain parameters ($\theta = 0.05$)

	Average	Upper bounds	Lower bounds	Errors (%)
$R(\lambda_{1,2})$	−0.50000	−0.48644	−0.51356	5.42326
$I(\lambda_{1,2})$	1.51022	1.54229	1.47815	4.24715
$R(\lambda_{3,4})$	−0.50000	−0.48644	−0.51356	5.42326
$I(\lambda_{3,4})$	0.46821	0.50148	0.43495	14.2102

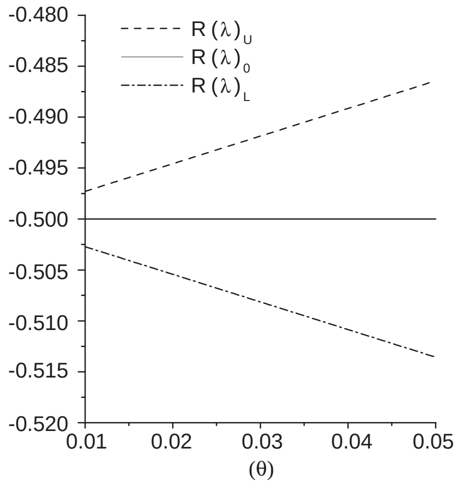

Fig. 2. The upper and lower bounds of the real part of the first and second eigenvalues.

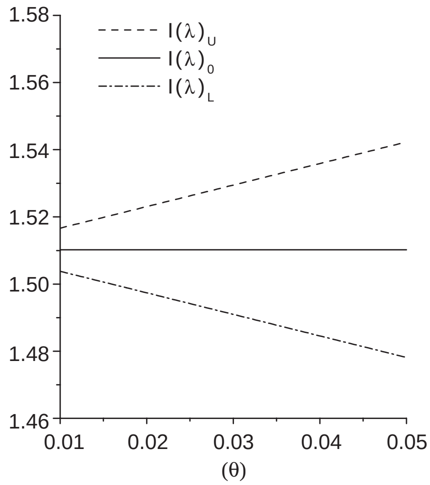

Fig. 3. The upper and lower bounds of the imaginary part of the first and second eigenvalues.

imaginary part of the third and fourth eigenvalues; if $\theta = 0.05$, the max relative error is 14.2102% at the imaginary part of the third and fourth eigenvalues.

7. Conclusions

The vibration control of structures with uncertain parameters is discussed in this paper. The control problem is approximated with the corresponding deterministic system. The uncertain parameters are modelled to be a convex elliptical set rather than a probabilistic set. This does not require the probabilistic distribution descriptions of the uncertain parameters. The formulas for

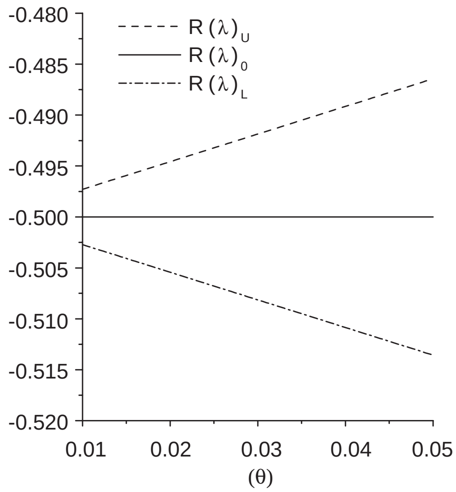

Fig. 4. The upper and lower bounds of the real part of the third and fourth eigenvalues.

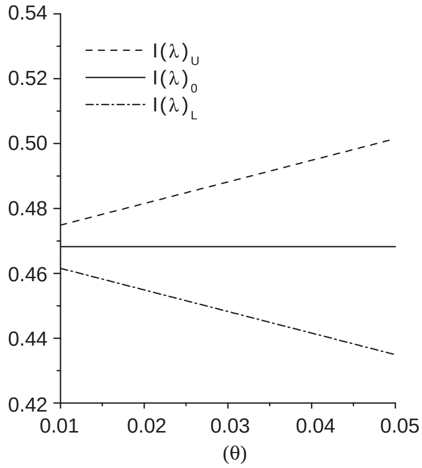

Fig. 5. The upper and lower bounds of the imaginary part of the third and fourth eigenvalues.

estimating the upper and lower bounds of eigenvalues of the closed-loop system are derived using the perturbation and optimal method. The results of the numerical example of a vibration system show that the upper and lower bounds of eigenvalues of the closed-loop system are proportional to θ and the method presented in this paper is effective for dealing with the vibration control of the uncertain systems. It should be noted that the present approach is limited to the case where the uncertainties of the parameters of the systems are small, because the higher order terms of Eq. (7) are neglected and the first order perturbation is used in Eq. (36). If the uncertainties of parameters of systems are fair large, the second order terms in Eq. (7) and the second order perturbation should be considered.

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